ON THE CRITICAL CASES OF STABILITY OF STATIONARY MOTIONS ACCORDING TO LIAPUNOV

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1. A system of differential equations

$$\frac{dx^i}{dt} = X^i(a, x) \qquad (i = 1, \ldots, n) \tag{1.1}$$

is considered. The functions $X^i(a, x)$ do not depend explicitly on t and are holomorphic with respect to the variables x^j in a sufficiently small neighborhood H of the equilibrium point $x^1 = \ldots = x^n = 0$, i.e.

$$X^i(a, x) = \sum_{s=1}^{\infty} q_s^i(a, x)$$

In the homogeneous forms of power $s \geqslant 1$

$$q_s{}^i(a, x) = a^i_{i_1 \ldots i_s} x^{i_1} \ldots x^{i_s}$$

the indices of summation $i_1, \ldots, i_s$ assume independently from each other all values between 1 and n. The coefficients $a^i_{i_1 \ldots i_s}$ are assumed to be undetermined and symmetric with respect to the lower indices.

Denote by m an arbitrary fixed natural number and introduce the notations

$$P_m{}^i(a, x) = \sum_{s=1}^{m} q_s^i(a, x), \qquad R_m{}^i(a, x) = \sum_{s=m+1}^{\infty} q_s^i(a, x)$$

In the paper certain propositions are formulated, concerning the conditions which the coefficients $a^i_{i_1}, \ldots, a^i_{i_1 \ldots i_m}$ of the polynomials $P^i_m(a, x)$ necessarily satisfy in the cases where the system of equations for the mth approximation

$$\frac{dx^i}{dt} = P_m^i(a, x) \tag{1.2}$$

does not permit us to solve the problem of stability of the trivial solution of system (1.1).

The idea of the paper consists of the following. System (1.1) is submitted to a general analytic transformation of coordinates, the Jacobian of which at the point $x = 0$ is different from zero. The solutions of system (1.1) and of the system obtained from it by means of such a transformation are, by virtue of the properties of the transformation, simultaneously stable or unstable. Therefore, in the critical cases, i.e. when system (1.2) does not permit us to solve the stability problem of the solutions of system (1.1), it is impossible to find an analytic transformation which implies that in the transformed system the coefficients of the functions $P_m^i(a, x)$ remain the same as before, while the coefficients of the functions $R_m^i(a, x)$ assume any values given in advance.

The conditions showing the impossibility of realizing the indicated transformation in the critical cases give the necessary conditions for the latter. The equalities expressing these conditions are the equations of the invariant varieties of a certain group of transformations of the coefficients a of the functions $P_m^i(a, x)$.

In the paper use is made of certain simple facts from the theory of continuous groups of transformations [1, 2].

2. All possible real values which the coefficients $a_{i_1}^i, \ldots, a_{i_1 \ldots i_m}^i$ can assume are considered as the coordinates of the points of a Euclidean space E_{N_m} of dimension N_m, equal to the common number of coefficients a in the functions $P_m^i(a, x)(i = 1, \ldots, n)$.

In the sequence $E_{N_1}, \ldots, E_{N_m}$ each space E_{N_σ} $(\sigma = 1, \ldots, m-1)$ is imbedded in any of the following spaces:

$$E_{N_1} \subset E_{N_2} \subset \ldots \subset E_{N_m}$$

and is its plane subspace of dimension N_σ. The space E_{N_m} is an N_m-dimensional extension of the space E_{N_σ}.

A point of the space E_{N_m} will be called non-critically stable (unstable) if its coordinates are such that the trivial solution of system (1.1) is stable (unstable) independently of the magnitudes of the coefficients $a_{i_1 \ldots i_{m+1}}^i, a_{i_1 \ldots i_{m+2}}^i, \ldots, a_{i_1 \ldots i_{m+\mu}}^i$ of the functions $R_m^i(a, x)$, where μ is a finite natural number.

Those points of the space E_{N_m}, for which by variation of the above coefficients it is possible to obtain according to one's wish stability or instability of the trivial solution of system (1.1), will be called critical points.

The sets of non-critical points of the space E_{N_m} are non-empty. This fact follows from a theorem by Krasovskii [3,4]. There exist also examples of non-empty sets of critical points.

Let us note the fact that the non-critical nature of the coefficients of system (1.2) has to do with its structural stability [4]. However, the assertion that the coefficients of a structurally stable system determine a point of the non-critical set holds only in the case $P^i_m(a, x) = q^i_m(a, x)$ for which the existence of the necessary conditions to be satisfied by the Liapunov function and its derivative have been proved [3,4].

Consider a general analytic transformation of coordinates

$$x^i = \varphi^i(x', \alpha) = \sum_{s=1}^{\infty} \alpha^i_{i_1 \ldots i_s} x'^{i_1} \ldots x'^{i_s} \tag{2.1}$$

$$\frac{\partial(\varphi^1, \ldots, \varphi^n)}{\partial(x'^1, \ldots, x'^n)}\bigg|_{x'=0} \equiv |\alpha^i_{i_1}| \neq 0 \tag{2.2}$$

By virtue of condition (2.2) there exists a neighborhood $h \subset H$ of the point $x = 0$ in which it realizes a one-to-one correspondence between the variables x^i and x'^i.

Consequently, for any finite value of the parameters a, subject to condition (2.2) only, the variables x^i and x'^i are equivalent with respect to stability in the sense of Liapunov. This is so because of the fact that stability is a local property of an equilibrium point [5,6].

Transforming system (1.1) by means of transformation (2.1) we obtain

$$\frac{dx^i}{dt} = X^i(a', x') \tag{2.3}$$

$$a'^j = f^j(a, \alpha) \tag{2.4}$$

where the $f^j(a, \alpha)$, as will be shown in Section 3, being integral rational functions of a and α, are such that $a'^i_{i_1}, \ldots, a'^i_{i_1 \ldots i_m}$ depend only on the quantities

$$a^i_{i_1}, \ldots, a^i_{i_1 \ldots i_m}, \qquad \alpha^i_{i_1}, \ldots, \alpha^i_{i_1 \ldots i_m}$$

Therefore, relations (2.4) represent transformations of each of the spaces E_{N_1}, E_{N_2}, ... into itself.

Let us prove that transformations (2.4) form a group.

The transformations $x'^i = \phi^i(x'', a')$ transform system (2.3) into the system

$$\frac{dx''^i}{dt} = X^i(a'', x''), \; a''^j = f^j(a', \alpha') = f^j[f^j(a, \alpha), \alpha'] \tag{2.5}$$

On the other hand, transforming Equations (1.1) directly by means of the transformation

$$x^i = \varphi^i(x', \alpha) = \varphi^i[\varphi^i(x'', \alpha'), \alpha] = \varphi^i[x'', \beta(\alpha, \alpha')]$$

we obtain the same system (2.5) for which

$$a''^j = f^j[a, \beta(\alpha, \alpha')]$$

Comparing the expressions of a''^j in the first and second case, and observing also the fact that analytic transformation (2.1) is reversible (and, consequently, transformations (2.4) are also reversible) and that for $a^j_i = \delta^j_i$, $a^j_{i_1 \ldots i_s} = 0$ $(s \geqslant 2)$ it coincides with the identical transformation, we convince ourselves of the correctness of the initial assertion.

By virtue of the fact that transformations (2.4) transform each of the spaces E_{N_1}, E_{N_2}, ... into itself, those which transform the points of the given space E_{N_m}, form a finite continuous group.

3. Let us find transformations (2.4). Substituting in Equations (1.1) the expressions of x^i given by (2.1) and those of $dx^i dt$ from (2.3) and equating among themselves the alternative coefficients of equal powers $x'^{\kappa_1} \ldots x'^{\kappa_k}$ entering on the left- and right-hand sides of the identities obtained, we have

$$\sum_{(\varkappa_1 \ldots \varkappa_k)} \Big(\sum_{\mu=1}^{k} \mu \alpha^j_{\delta \varkappa_1 \ldots \varkappa_{\mu-1}} a'^{\delta}_{\varkappa_\mu \ldots \varkappa_k} - $$
$$- \sum_{s=1}^{k} \sum_{\mu_1 + \ldots + \mu_s = k} a^j_{i_1 \ldots i_s} \alpha^{i_1}_{\varkappa_1 \ldots \varkappa_{\mu_1}} \ldots \alpha^{i_s}_{\varkappa_{\mu_1 + \ldots + \mu_{s-1}+1} \ldots \varkappa_k} \Big) = 0 \tag{3.1}$$
$$(j, \delta, \varkappa, i = 1, 2, \ldots, n; \; k = 1, 2, \ldots)$$

Here, and in what follows, indices in parentheses ($\kappa_1, \ldots, \kappa_k$) under the summation sign denote summation over all possible permutations of the indices $\kappa_1, \ldots, \kappa_k$.

Determine the quantities $a_j^{*\epsilon}$ by means of the relations

$$\alpha_\alpha{}^j \alpha_j{}^{*\varepsilon} = \delta_\alpha{}^\varepsilon$$

where $\delta_\alpha{}^\epsilon$ is the Kronecker symbol.

By virtue of (2.2) relations (3.1) can be uniquely solved for $a'^{\epsilon}_{\kappa_1\ldots\kappa_k}$. We then obtain

$$a'^{\varepsilon}_{\varkappa_1\ldots\varkappa_k} = \sum_{(\varkappa_1\ldots\varkappa_k)} \Big(\sum_{s=1}^{k} \sum_{\mu_1+\ldots+\mu_s=k} a^j_{i_1\ldots i_s} \alpha^{i_1}_{\varkappa_1\ldots\varkappa_{\mu_1}} \cdots \alpha^{i_s}_{\varkappa_{\mu_1+\ldots+\mu_{s-1}+1}\ldots\varkappa_k} - \sum_{\mu=2}^{k} \mu a'^{\delta}_{\varkappa_\mu\ldots\varkappa_k} \alpha^j_{\delta\varkappa_1\ldots\varkappa_{\mu-1}}\Big) \alpha_j^{*\varepsilon} \qquad (3.2)$$

so that transformations (2.4) can be written in the form of the following sequence:

$$k = 1, \qquad a'^{\varepsilon}_{\varkappa_1} = a^j_{i_1} \alpha^{i_1}_{\varkappa_1} \alpha_j^{*\varepsilon}$$

$$k = 2, \qquad a'^{\varepsilon}_{\varkappa_1\varkappa_2} = (a^j_{i_1 i_2} \alpha^{i_1}_{\varkappa_1} \alpha^{i_2}_{\varkappa_2} + a^j_{i_1} \alpha^{i_1}_{\varkappa_1\varkappa_2} - a'^{\delta}_{\varkappa_1} \alpha^j_{\delta\varkappa_2} - a'^{\delta}_{\varkappa_2} \alpha^j_{\delta\varkappa_1}) \alpha_j^{*\varepsilon}, \ldots$$

From here, in particular, the reversibility of transformation (2.4) is seen.

Denoting by $a^1, \ldots, a^{N_m}$ and $\alpha^1, \ldots, \alpha^{N_m}$ one-dimensional ordered systems of quantities $a^j_{i_1}, \ldots, a^j_{i_1\ldots i_m}$ and $\alpha^j_{i_1}, \ldots, \alpha^j_{i_1\ldots i_m}$, let us find the components $\xi^i_b(a)$ of the vector matrix which determines the infinitesimal operators

$$X_b f = \xi^i_b(a) \frac{\partial f}{\partial a^i} \qquad (i,\ b = 1, \ldots, N_m)$$

of the group of transformations of the space E_{N_m}.

Having agreed upon a certain definite order of enumeration of the quantities a and α, the same for both sequences, we shall evaluate the quantities

$$\xi^i_b(a) = \left(\frac{\partial a'^i}{\partial \alpha^b}\right)_{\alpha=\alpha^\circ}$$

for $\alpha_i^{\circ j} = \delta^j_i$, $\alpha^{\circ j}_{i_1 i_2} = \alpha^{\circ j}_{i_1 i_2 i_3} = \ldots = 0$. Differentiating (3.1) with respect to $\alpha^\alpha_{\gamma_1\ldots\gamma_l}$ and setting then α equal to the indicated particular values, we obtain

$$\left(\frac{\partial a'^{j}_{\varkappa_1\ldots\varkappa_k}}{\partial \alpha^{\alpha}_{\gamma_1\ldots\gamma_l}}\right)_{\alpha=\alpha^\circ} = \tag{3.3}$$

$$= \begin{cases} \frac{1}{k!}\sum\limits_{(\varkappa_1\ldots\varkappa_l)} \left[(k-l+1)\, a^{j}_{\alpha\varkappa_1\ldots\varkappa_{k-l}}\delta^{\gamma_1\ldots\gamma_l}_{\varkappa_{k-l+1}\ldots\varkappa_k} - l a^{\delta}_{\varkappa_l\ldots\varkappa_k}\delta^{j}_{\alpha}\delta^{\gamma_1\ldots\gamma_l}_{\delta\varkappa_1\ldots\varkappa_{l-1}}\right] & \text{for } k \geqslant l \\ 0 & \text{for } k < l \end{cases}$$

Here $\delta^{\gamma_1\ldots\gamma_l}_{\beta_1\ldots\beta_l}$ denotes the Kronecker tensor, taken in modulus.

The vector matrix of the group of transformations of the space E_{N_m} can be written in the block form

$$\mathrm{M}_m = \left\| \begin{matrix} (a^{j}_{i_1})^1_1 & (a^{j}_{i_1 i_2})^2_1 & \ldots & (a^{j}_{i_1\ldots i_m})^m_1 \\ 0 & (a^{j}_{i_1})^2_2 & \ldots & (a^{j}_{i_1\ldots i_{m-1}})^m_2 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & (a^{j}_{i_1})^m_m \end{matrix} \right\|$$

where the block

$$(a^{j}_{i_1\ldots i_{k-l+1}})^k_l = \left\| \left(\frac{\partial a'^{j}_{\varkappa_1\ldots\varkappa_k}}{\partial \alpha^{\alpha}_{\gamma_1\ldots\gamma_l}}\right)_{\alpha=\alpha^\circ} \right\| \qquad (l \leqslant k \leqslant m)$$

is a matrix with elements depending (linearly) on the quantities a, having exactly $k - l + 1$ lower indices.

The vector matrix M_σ of the group of transformations of the space E_{N_σ} ($\sigma \leqslant m$) is obtained from M_m by singling out the principal diagonal matrix of order N_σ.

Let us prove that for $\sigma \geqslant 2$ the order of the group of transformations of the space E_{N_σ} is equal to N_σ. For this purpose it is sufficient to make sure that the matrix M_m does not contain linearly dependent rows with constant coefficients. Let us prove this first for the matrix

$$\left\| (a^{j}_{i_1})^1_1 \quad (a^{j}_{i_1 i_2})^2_1 \ldots (a^{j}_{i_1\ldots i_m})^m_1 \right\| \tag{3.4}$$

In fact, at the point $a^{j}_{i_1 i_2} = \delta^{ij}_{i_1 i_2}$ the components of its block $(a^{j}_{i_1 i_2})^2_1$ have the form

$$\left(\frac{\partial a'^{j}_{\varkappa_1\varkappa_2}}{\partial \alpha^{\alpha}_{\gamma}}\right)_{\alpha=\alpha^\circ} = \delta^{jj}_{\alpha\varkappa_1}\delta^{\gamma}_{\varkappa_2} + \delta^{jj}_{\alpha\varkappa_2}\delta^{\gamma}_{\varkappa_1} - \delta^{\gamma\gamma}_{\varkappa_1\varkappa_2}\delta^{j}_{\alpha}$$

Selecting those columns for which $\kappa_2 = j$ and taking into account

$$\delta^{\gamma\gamma}_{\varkappa_1\varkappa_2} = \delta^{(\gamma)}_{\varkappa_1}\delta^{(\gamma)}_{\varkappa_2} = \delta^{(\gamma)}_{\varkappa_1}\delta^{(\gamma)}_{j}$$

(summation according to the index in parentheses is not carried out) let us construct a quadratic diagonal matrix with determinant of order N_1, namely

$$\left|\left(\frac{\partial a'^{j}_{\varkappa_1 j}}{\partial \alpha^{\alpha}_{\gamma}}\right)_{\alpha=\alpha^\circ}\right| = |\delta^{(j)}_{\alpha}\delta^{(j)}_{\varkappa_1}| = 1$$

Therefore, the rows of matrix (3.4) are linearly independent. In relation (3.3) put $k = m = l$, $a^j_{i_1} = \delta^j_{i_1}$. We then obtain

$$\left(\frac{\partial a'^{j}_{\varkappa_1\ldots\varkappa_l}}{\partial \alpha^{\alpha}_{\gamma_1\ldots\gamma_l}}\right)_{\alpha=\alpha^\circ} = \frac{1}{l!}\sum_{(\varkappa_1\ldots\varkappa_l)}\left(\delta^{j}_{\alpha}\delta^{\gamma_1\ldots\gamma_l}_{\varkappa_1\ldots\varkappa_l} - l\delta^{\delta}_{\varkappa_l}\delta^{j}_{\alpha}\delta^{\gamma_1\ldots\gamma_l}_{\delta\varkappa_1\ldots\varkappa_{l-1}}\right)$$
$$= \frac{1-l}{l!}\sum_{(\varkappa_1\ldots\varkappa_l)}\delta^{j}_{\alpha}\delta^{\gamma_1\ldots\gamma_l}_{\varkappa_1\ldots\varkappa_l} = (1-l)\,\delta^{j}_{\alpha}\delta^{\gamma_1\ldots\gamma_l}_{\varkappa_1\ldots\varkappa_l}$$

By virtue of the above method of enumeration of the sequences a and α the matrix

$$\left\| (1-l)\,\delta^{j}_{\alpha}\delta^{\gamma_1\ldots\gamma_l}_{\varkappa_1\ldots\varkappa_l} \right\|$$

is a diagonal matrix and its determinant is

$$|(a^j_{i_1})^l_l| = (1-l)^{N_l - N_{l-1}} \neq 0 \qquad (l \neq 1) \tag{3.5}$$

Therefore the rank of the matrix

$$\left\|\begin{matrix} (a^j_{i_1})^2_2 & (a^j_{i_1 i_2})^3_2 & \ldots & (a^{j1}_{i_1\ldots i_{m-1}})^m_2 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & (a^j_{i_1})^m_m \end{matrix}\right\|$$

is equal to $N_m - N_1$.

It is clear that none of the rows of matrix (3.6) can be a linear combination with constant coefficients of the rows of matrix (3.4), and, conversely, because such a combination would connect quantities a with different numbers of lower indices which is impossible.

Thus, the points of the space E_{N_m} are subject to the group of transformations G_{N_m} $(m \neq 1)$ of order N_m.

It can be shown that for $m = 1$ we have the group $G_r (r < N_1)$.

4. Let $b_1'^{i_1} = f_1^{i_1}(b_1, \beta_1)$ be transformations of the space E_{n_1} which form the group G_{n_1} with infinitesimal operators

$$X_{l_1}^1 f = \xi_{1l_1}^{i_1}(b_1)\frac{\partial f}{\partial b_1^{i_1}}, \qquad \xi_{1l_1}^{i_1}(b_1) = \left(\frac{\partial f_1^{i_1}}{\partial \beta_1^{l_1}}\right)_{\beta_1=\beta_1^\circ} \qquad (l_1,\ i_1 = 1,\ \ldots,\ n_1)$$

Further, let the transformations of the space $E_{n_2} (n_2 > n_1)$

$$b_1'^{i_1} = f_1^{i_1}(b_1,\ \beta_1) \qquad b_2'^{i_2} = f_2^{i_2}(b_1,\ b_2,\ \beta_1,\ \beta_2) \qquad \begin{pmatrix} i_1 = 1,\ \ldots,\ n_1 \\ i_2 = 1,\ \ldots,\ n_2 - n_1 \end{pmatrix}$$

form a group G_{n_2} with operators

$$\begin{aligned} X_{l_1}^2 f &= \xi_{1l_1}^{i_1}(b_1)\frac{\partial f}{\partial b_1^{i_1}} + \xi_{2l_1}^{i_2}(b_2)\frac{\partial f}{\partial b_2^{i_2}} \\ X_{l_2}^2 f &= \qquad\qquad\qquad \xi_{2l_2}^{i_2}(b_1)\frac{\partial f}{\partial b_2^{i_2}} \end{aligned} \qquad (i_2,\ l_2 = 1,\ \ldots,\ n_2 - n_1)$$

$$\xi_{2l_1}^{i_2}(b_2) = \left(\frac{\partial f_2^{i_2}}{\partial \beta_1^{l_1}}\right)_{\beta=\beta^\circ}, \qquad \xi_{2l_2}^{i_2}(b_1) = \left(\frac{\partial f_2^{i_2}}{\partial \beta_2^{l_2}}\right)_{\beta=\beta^\circ} \qquad (4.1)$$

To the values $\beta = \beta^\circ$ there corresponds the identical transformation of both groups.

Further, let

$$|\xi_{2l_2}^{i_2}(b_1)| \not\equiv 0 \qquad (4.2)$$

and let the common rank of the matrix $\|\xi_{1l_1}^{i_1}\|$ be equal to $r < n_1$. Obviously, not all minors of rank $n_2 - n_1 + q (q \leqslant r)$ of the matrix

$$\mathrm{M}(b_1,\ b_2) = \begin{Vmatrix} \xi_{1l_1}^{i_1}(b_1) & \xi_{2l_1}^{i_2}(b_2) \\ 0 & \xi_{2l_2}^{i_2}(b_1) \end{Vmatrix}$$

are identically zero. For arbitrary $q \leqslant r$ there will be such among them which depend only on the quantity b_1; for example, all diagonal minors are of this nature. Therefore, the complete system of relations, by virtue of which all minors of order $n_2 - n_1 + q$ of the matrix $M(b_1, b_2)$ vanish, can be given in the form

$$\begin{array}{c} \varphi^q_{11}(b_1) = \ldots = \varphi^q_{1\rho_1}(b_1) = \varphi^q_{1,\rho_1+1}(b_1, b_2) = \ldots = \varphi^q_{1\tau_1}(b_1, b_2) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \varphi^q_{\pi 1}(b_1) = \ldots = \varphi^q_{\pi\rho_\pi}(b_1) = \varphi^q_{\pi,\rho_\pi+1}(b_1, b_2) = \ldots = \varphi^q_{\pi\tau_\pi}(b_1, b_2) = 0 \end{array} \tag{4.3}$$

Here ϕ are independent integral rational functions of their arguments. Each chain of equalities determines a certain irreducible simply connected manifold of dimension $n_2 - r$ which is an invariant manifold of the group G_{n_2}. It consists of singular points of order not higher than $n_2 - n_1 + q + 1$. The number of such manifolds is π. It is assumed that of all the possible analytic representations of each of the irreducible manifolds that one is chosen for which the corresponding number ρ is maximum. The following lemma will be proved.

Lemma. Any point $\overset{*}{b}_1(\overset{*}{b}{}_1^1, \ldots, \overset{*}{b}{}_1^{n_1})$ of the space E_{n_1}, ordinary $(q = r)$ or singular of order $q < r$, if it is such that there does not exist a selection of parameters β_1^j, β_2^j leading to the realization of the transformation

$$\overset{*}{b}{}_1^{i_1} = f_1^{i_1}(\overset{*}{b}_1, \beta_1), \qquad b_2'^{i_2} = f_2^{i_2}(\overset{*}{b}_1, b_2, \beta_1, \beta_2) \tag{4.4}$$

where all the $b_2'^{i_2}$ are certain numbers given in advance, then the coordinates of the point $\overset{*}{b}_1$ necessarily satisfy at least one of the relations

$$\begin{array}{c} \varphi^q_{11}(b_1) = \ldots = \varphi^q_{1\rho_1}(b_1) = 0 \\ \cdots\cdots\cdots\cdots \\ \varphi^q_{\pi 1}(b_1) = \ldots = \varphi^q_{\pi\rho_\pi}(b_1) = 0 \end{array} \tag{4.5}$$

Proof. Construct for the point $\overset{*}{b}_1$, assumed to be singular and of order q, a stationary subgroup $H_{n_1 - q}$ of the group G_{n_1}. Then the finite transformations of this subgroup

$$b_1'^{i_1} = f_1^{i_1}[b_1, \beta_1(u)] \qquad (\overset{*}{b}{}_1^{i_1} = f_1^{i_1}[\overset{*}{b}_1, \beta_1(u)])$$

contain $n_1 - q$ essential parameters $u^1, \ldots, u^{n_1 - q}$. To the values $u^\nu = \overset{\circ}{u}{}^\nu(\overset{\circ}{\beta}{}_1^\mu = \beta_1^\mu(u^\circ))$ there corresponds the identical transformation.

Set

$$c_\nu{}^{l_1} = \left(\frac{\partial \beta_1{}^{l_1}}{\partial u^\nu}\right)_{u=u^\circ}$$

The rank of the matrix $\| c_\nu^{l_1} \|$ is equal to $n_1 - q$. Without restricting generality we can write

$$\| c_\nu^{l_1} \| = \begin{Vmatrix} c_1^1 & \ldots c_1^q & 1 \ldots 0 \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ c_{n_1-q}^1 & \ldots c_{n_1-q}^q & 0 \ldots 1 \end{Vmatrix}$$

In the opposite case it is sufficient to replace the parameters $\beta_1^{q+1}, \beta_1^{q+2}, \ldots, \beta_1^{n_1}$ of the group G_{n_1} by new parameters according to the formulas

$$\beta_1^{q+1} = \beta_1^{q+1}(u), \ldots, \beta_1^{n_1} = \beta_1^{n_1}(u)\cdot$$

This is possible because we can assume without restricting generality that

$$\frac{\partial (\beta_1^{q+1}, \ldots, \beta_1^{n_1})}{\partial (u^1, \ldots, u^{n_1-q})} \neq 0$$

Adding to the μth row of the matrix $M(b_1, b_2)(q < \mu \leqslant n_1)$ the rows with indices 1, 2, ..., q, after having them multiplied by $c^1_{\mu-q}, \ldots, c^q_{\mu-q}$ respectively, we obtain

$$M(b_1, b_2) = \left\| \begin{array}{c:c} \xi_{1\varepsilon}^{i_1}(b_1) & \xi_{2\varepsilon}^{i_2}(b_2) \\ \hdashline \xi_{1\mu}^{i_1}(b_1) + c_{\mu-q}^{\omega}\xi_{1\omega}^{i_1}(b_1) & \\ \hdashline 0 & M_* \end{array} \right\| \qquad \begin{pmatrix} \omega, \varepsilon = 1, \ldots, q \\ \mu = q+1, \ldots, n_1 \end{pmatrix}$$

By virtue of (4.2) and

$$\left(\frac{\partial f_2'^{i_2}}{\partial u^\nu}\right)_{u=u^\circ} = c_\nu^{l_1}\xi_{2l_1}^{i_2}(b_2) = \xi_{2\mu}^{i_2}(b_2) + c_{\mu-q}^{\omega}\xi_{2\omega}^{i_2}(b_2)$$
$$(\nu = 1, \ldots, n_1 - q)$$

the matrix M_* is of the form

$$M_* = \begin{Vmatrix} \xi_{2\mu}^{i_2}(b_2) + c_{\mu-q}^{\omega}\xi_{2\omega}^{i_1}(b_2) \\ \xi_{2l_2}^{i_2}(b_1) \end{Vmatrix} = \begin{Vmatrix} \left(\dfrac{\partial f_2^{i_2}}{\partial u^\nu}\right)_{u=u^\circ} \\ \left(\dfrac{\partial f_2^{i_2}}{\partial \beta_2^{l_2}}\right)_{\beta_2=\beta_2^\circ} \end{Vmatrix}$$

Because the subgroup H_{n_1-q} is stationary we have

$$\xi_{1\mu}^{i_1}(b_1^*) + c_{\mu-q}^{\omega}\xi_{1\omega}^{i_1}(b_1^*) = 0\,.$$

and, consequently, at the point $b_1 = b_1^*$ the matrix $M(b_1, b_2)$ assumes the form

$$M(b_1^*, b_2) = \left\| \begin{array}{c|c} \xi_{1\varepsilon}^{i_1}(b_1^*) & \xi_{2\varepsilon}^{i_2}(b_2) \\ \hline 0 & \\ \cline{1-1} 0 & M_* \end{array} \right\|$$

Assume that the point b_1^* satisfied the conditions of the lemma but its coordinates do not satisfy any of the relations (4.5). Then among the minors of order $n_2 - n_1 + q$ of the matrix $M(b_1^*, b_2)$ there will be one which is different from zero. Because the rank of the matrix $\|\xi_{1\epsilon}^{i_1}(b_1^*)\|$ is equal to q, there exists for arbitrary values of b_2^j a minor of order $n_2 - n_1$ of the matrix M_* which is not zero. The vector matrix of the group G_{n_2} is the matrix

$$\left\| \begin{array}{cc} \partial f_1^{i_1}/\partial\beta_1^{e_1} & \partial f_2^{i_2}/\partial\beta_1^{l_1} \\ 0 & \partial f_2^{i_2}/\partial\beta_2^{l_2} \end{array} \right\|$$

evaluated for $\beta_1^{i_1} = \overset{\circ}{\beta}{}_1^{i_1}$ $\beta_2^{i_2} = \overset{\circ}{\beta}{}_2^{i_2}$. Assigning to the quantities $\beta_1^{i_1}$, $\beta_2^{i_2}$ other fixed values we obtain again a matrix of the group G_{n_2} which can be formed by multiplying the first one by a certain constant non-degenerate matrix [1]. As a consequence of this fact the ranks of both matrices at any given point (b_1, b_2) are equal [7].

Therefore the rank of the matrix

$$\left\| \begin{array}{c} \partial f_2^{i_2}/\partial u^{\nu} \\ \partial f_2^{i_2}/\partial\beta_2^{l_2} \end{array} \right\|$$

also does not change if it is evaluated for values of u^{ν}, $\beta_2^{l_2}$ different from the primary ones.

Then by a proper choice of values for u and β_2 the quantities $b_2'^{i_2}$ in transformations (4.4) can be made to assume desired values. This, however, contradicts the assumption of the lemma. This completes the proof of the lemma.

Let us note that relations (4.5) are the equations of the invariant varieties of the group G_{n_1}.

In fact, for $t = 1, \ldots, \rho_l (l < \pi)$ the operators

$$X^2_{l_1}\varphi^q_{lt}(b_1) \equiv X^1_{l_1}\varphi^q_{lt}(b_1)$$

can vanish identically or by virtue of those relations (4.4), corresponding to the given l, which depend only on b_1, i.e. by virtue of the equalities

$$\varphi^q_{l1}(b_1) = \ldots = \varphi^q_{l\rho_l}(b_1) = 0$$

Introduce the following definition.

The order of a critical point of the space E_{N_m} will be defined as a natural number $p_m \geq m + 1$ for which:

1) stability (instability) of the trivial solution of system (1.1) is preserved no matter what are the coefficients a of the forms $q^i_\mu(a, x)$ $(\mu = m + 1, \ldots, p_m - 1)$;

2) by a change of the coefficients of the forms $q^i_{p_m}(a, x)$ stability or instability of the solutions of system (1.1) can be obtained according choice. Obviously, any critical point of the space E_{N_m} has a finite order; in the opposite case such a point is not critical at all (in the sense of the definition given in Section 2).

Everywhere in what follows we shall consider only those points of the space E_{N_m} which belong to a closed spherical neighborhood of the origin of the coordinates with an arbitrarily large but finite radius. Consequently, we can indicate for the magnitudes of the orders of the critical points belonging to this neighborhood the exact upper bound P_m because a function, bounded at every point of a closed set, is bounded in this set.

Assume that the space E_{N_m} consists of ordinary points $(q = r)$ and singular points of orders $0 \leq q_1 \leq \ldots \leq q_l < r$, and let P_m be the highest order of critical points contained in it. Let us find the relations to be satisfied by the coordinates of the critical points of the space E_{N_m}.

Consider the matrix M_h of the group $G_{N_h} (h = P_m)$. It is not difficult to indicate its minors of order $N_h - N_m + q_\epsilon (\epsilon = 1, \ldots, e)$ which are different from zero and depend only on the quantities $a^j_{i_1}, \ldots, a^j_{i_1 \ldots i_m}$. The relations, by virtue of which all minors of order $N_h - N_m + q_\epsilon$ of the matrix M_h vanish, can be written in the form of relations (4.3) where b^j_1, b^j_2 are linearly ordered sets of the quantities $a^j_{i_1}, \ldots, a^j_{i_1 \ldots i_m}$ and $a^j_{i_1 \ldots i_{m+1}}, \ldots, a^j_{i_1 \ldots i_h}$. Let $n_2 = N_h$, $n_1 = N_m$, $q_\epsilon = q$.

Making use of the lemma we can convince ourselves immediately of the correctness of the following theorem.

Theorem. If the point $(a^j_{i_1}, \ldots, a^j_{i_1 \ldots i_m})$ of the space E_{N_m} is a critical point, then its coordinates satisfy necessarily at least one of the relations (4.5). The manifolds they determine are invariant manifolds of the group G_{N_m}.

The author has considered a system of the second order ($n = 2$). For the system of the first approximation the known critical manifolds were obtained. The critical case of a single zero root was discussed. Specifying the coefficients by the equalities $a^1_1 = a^1_2 = 0$, $a^2_1 = k$, $a^2_2 = -1$ for systems of second and third approximations, the critical manifolds coincide with the boundaries of the region of stability in a well-known example of Liapunov [5].

5. Let us find out to what extent the method used in the previous sections reflects the specific nature of the stability problem.

Let the system of equations

$$\Psi^i(x, a) = 0 \qquad (i = 1, \ldots, n) \qquad (x \subset \mathrm{H}) \tag{5.1}$$

be given, where the Ψ^i stand for the symbols of algebraic or differential and algebraic operations on the variables $x^1, \ldots, x^n$ with coefficients $a^1, \ldots, a^N$ which can assume all possible real values.

Assume that it is required to find necessary and sufficient conditions in order that a certain property (ζ) of the solutions of system (5.1) be realized for $x \subset H$.

It is obvious that if the form of the operator Ψ^i is given, then the necessary and sufficient conditions for the realization of a certain property of the solutions of Equations (5.1) depend on the coefficients a only.

These conditions are covariant with respect to arbitrary transformations of Equations (5.1) which preserve the property sought and the form of the equations.

Let $x^j = \phi^j(x', a)$ be such transformations which form an r-parameter group. Then

$$\begin{gathered}\Psi^i(x, a) = \eta^i_\varkappa(x', \alpha)\, \Psi^\varkappa(x', a') \\ |\eta^i_\varkappa(x', \alpha)| \neq 0 \quad (x' \subset \mathrm{H}), \qquad a'^\varepsilon = f^\varepsilon(a, \alpha) \qquad \begin{pmatrix}\varkappa = 1, \ldots, n \\ \varepsilon = 1, \ldots, N\end{pmatrix}\end{gathered} \tag{5.2}$$

and transformations (5.2) form a group $G_\rho(\rho < r)$. The regions of the space E_N of the coefficients a in which the property (ζ) is realized, or the opposite property $(\bar{\zeta})$, if such property exists, are separated by the invariant manifolds of the group G_ρ.

If the group G_ρ is multiply transitive so many times that all its invariant manifolds, obtained by equating to zero the minors of the matrix of the group, are the minimum invariant manifolds for any of its points [1], then the regions sought exist and form systems of intransitivity of the group G_ρ which can be easily found.

For the completion of the problem it is now sufficient to verify that the property (ζ) is satisfied for any single point of each system of intransitivity.

Consider a simple example. Let it be required to find necessary and sufficient conditions in order that the quadratic form

$$\Psi(a, x) = a_{i_1 i_2} x^{i_1} x^{i_2} \qquad (i_1, i_2 = 1, 2)$$

be of definite sign.

The transformation $x^i = \alpha^i_j x'^j$ possesses the required properties. We have

$$\Psi(a, x) \equiv \Psi(a', x'), \qquad a'_{i_1 i_2} = a_{j_1 j_2} \alpha^{j_1}_{i_2} \alpha^{j_2}_{i_2}$$

The matrix of the group with components

$$\left(\frac{\partial a'_{i_1 i_2}}{\partial \alpha_\beta{}^\alpha}\right)_{\alpha^\varepsilon_\varkappa = \delta_\varkappa{}^\varepsilon} = a_{\alpha i_2} \delta^\beta_{i_1} + a_{\alpha i_1} \delta^\beta_{i_2}$$

has the form

$$\left\| \begin{matrix} a_{11} & a_{12} & 0 \\ 0 & a_{11} & a_{12} \\ a_{12} & a_{22} & 0 \\ 0 & a_{12} & a_{22} \end{matrix} \right\|$$

The group G_4 is simply transitive. The minors of the third order of the matrix vanish on the invariant manifold

$$a_{11} a_{22} - a_{12}{}^2 = 0$$

consisting of singular points of second order (excluding the point $a_{11} = a_{12} = a_{22} = 0$ which is a singular point of zero order). The regions $a_{11}a_{22} - a_{12}^2 > 0$ and $a_{11}a_{22} - a_{12}^2 < 0$ form systems of intransitivity and consist of ordinary points of the transformation.

It is obvious that for $a_{11} = a_{22} = 1$, $a_{12} = 0$ the form $\Psi(a, x)$ is of definite sign. Consequently, it is of definite sign also at any point of the region $a_{11}a_{22} - a_{12}^2 > 0$.

The regions in which the form $\Psi(a, x)$ is of variable or constant sign are defined analogously.

As seen from the above, the information about the specific nature of the property (ζ) (definiteness of the sign) was needed only at the last stage of the process of solving the problem. Its extent was insignificant.

If the group is intransitive, then the extent of the supplementary information necessary to solve the problem is possibly larger. Such information can often be obtained from Equations (5.1), and the equations themselves can be generalized in such a manner that they admit a group of higher order.

Thus, the equation

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = 0 \qquad (a_\nu = \text{const})$$

does not admit other transformations besides the identity transformation. The problem of stability of the trivial solution of this equation, however, can be generalized and solved completely, using the formulation described in previous sections.

In Section 4, additional information concerning stability was used and, consequently, the methods used there are specific for that type of problem.

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